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LETTER TO THE EDITOR

Real Planck distribution for a complex Q-boson gas*

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Abstract. We discuss the energy-density distribution for a gas of q-bosons with complex deformation parameter by exploiting a new q-complex oscillator algebra in which both the number operator and the energy eigenvalues are real. The corresponding Planck distribution generalizes the results obtained for a real q-boson gas, including Wien's and Stephan's laws, and the role of an effective Planck constant depending on the parameter modulus.

One of the most interesting issues in the theory of quantum deformations of Lie algebras and groups [1] is the study of the physical implications of the q-structures. For instance, the statistical properties of a gas of q-boson oscillators have been investigated by some authors [2–4]. In particular, recently, Gupta *et al* [5] have studied the q-deformation of the Planck energy-density distribution of a boson gas when the parameter q is complex, i.e. $q = q_1 + iq_2$, on the basis of [6–8].

Their starting point was the usual expression of the energy eigenvalues $E(q, n)$ for a q-harmonic oscillator, with the zero-point energy subtracted, i.e. [9]

$$E(q,n) = \tfrac{1}{2}\hbar\omega([n] + [n+1] - 1). \qquad (1)$$

The square bracket in (1) introduces the parameter q (or s) of the $SU_q(2)$ algebra according to the definition [5]

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{e^{sx} - e^{-sx}}{e^s - e^{-s}} = \frac{\sinh sx}{\sinh s} \qquad (2)$$

where $q = e^s$, $s = a + ib$ and a, b are real numbers. From the above definition, we see that when q is complex the energy eigenvalues also become complex. However, in our opinion, the results obtained in [5], by trivially generalizing equation (1) to the complex case, have no meaning because the energy-eigenvalue's form (1) is no longer valid for complex q.

Indeed, De Falco *et al* [10] have recently investigated the q-boson algebra with complex deformation parameter by using the new commutation relations

$$AB - qBA = F(q, N, \ldots) \qquad (3)$$

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and

$$B^+A^+ - q^*A^+B^+ = F^*(q^*, N, \ldots) \tag{4}$$

where A, B are non-Hermitian operators, $q \in C$ (C complex field) and N is the usual number operator ($N|n\rangle = n|n\rangle$).

For the case where q is complex, we have $B \neq A^+$. By using the bosonization method (originally introduced by Jannussis and co-workers [11]) for the special case $F(q, N, \ldots) = q^{-N}$ [10], we obtain the following representation:

$$A = \left(\frac{q^{(N+1)} - q^{-(N+1)}}{q - q^{-1}} \frac{1}{N+1}\right)^{1/2} a = \left(\frac{[N+1]}{N+1}\right)^{1/2} a \tag{5}$$

$$B = a^+ \left(\frac{[N+1]}{N+1}\right)^{1/2} \tag{6}$$

where the symbol $[x]$ is given by equation (2).

The new generalized number operator $\mathcal{N}$ is defined in [10] and has the form

$$\mathcal{N} = A^+A = BB^+ = ([N][N]^*)^{1/2} \tag{7}$$

with action on the kets $|n\rangle$ given by

$$\mathcal{N}|n\rangle = \left(\frac{q^n - q^{-n}}{q - q^{-1}} \frac{q^{*n} - q^{*-n}}{q^* - q^{*-1}}\right)^{1/2} |n\rangle = \left(\frac{|q|^{2n} + |q|^{-2n} - 2\cos(2n\theta)}{|q|^2 + |q|^{-2} - 2\cos 2\theta}\right)^{1/2} |n\rangle \tag{8}$$

where

$$q = |q|e^{i\theta} = q_1 + iq_2 \qquad \theta = \tan^{-1}\frac{q_2}{q_1}. \tag{9}$$

In the usual Fock representation of the Hamiltonian operator

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 x^2 = \frac{\hbar\omega}{2}(AA^+ + A^+A)$$

we get the following deformed spectrum:

$$E_n = \frac{\hbar\omega}{2}(|n+1| + |n|) = \frac{\hbar\omega}{2}\left[\left(\frac{|q|^{2(n+1)} + |q|^{-2(n+1)} - 2\cos[2\theta(n+1)]}{|q|^2 + |q|^{-2} - 2\cos 2\theta}\right)^{1/2} + \left(\frac{|q|^{2n} + |q|^{-2n} - 2\cos(2\theta n)}{|q|^2 + |q|^{-2} - 2\cos 2\theta}\right)^{1/2}\right] \tag{10}$$

i.e. the energy eigenvalues are real and not complex as in [5]. Nevertheless, the spectrum (10) mantains a remarkable richness of possible different cases, which are interesting from the theoretical point of view and deserve experimental verification. For instance, in the case $|q| = 1$, equation (10) takes the form

$$E_n = \frac{\hbar\omega}{2}\frac{\sin(n+1)\theta + \sin(n\theta)}{\sin\theta} = \frac{\hbar\omega}{2}\frac{\sin(n+1/2)\theta}{\sin\theta/2}. \tag{11}$$

Let us stress again that both the new number operator (9) and the energy spectrum (10) are real and not complex as in [5]. The mere substitution of the real parameter q with complex q in the formulae, valid for the usual deformed harmonic oscillator, is not, in our opinion, a correct procedure.

We now want to apply our results to obtaining the Planck distribution for a complex q-boson gas, thus generalizing Martin-Delgado's findings [3].

The energy eigenvalues (10) for a q-complex harmonic oscillator with zero-point energy subtraction are

$$\mathcal{E}(q,n) = \frac{\hbar\omega}{2}\left[\left(\frac{|q|^{(2n+1)} + |q|^{-2(n+1)} - 2\cos 2\theta(n+1)}{|q|^2 + |q|^{-2} - 2\cos 2\theta}\right)^{1/2} + \left(\frac{|q|^{2n} + |q|^{-2n} - 2\cos(2\theta\eta)}{|q|^2 + |q|^{-2} - 2\cos 2\theta}\right)^{1/2} - 1\right]. \tag{12}$$

The partition function $Z_q(\omega, T)$ for a q-boson gas in the canonical ensemble is

$$Z_q(\omega, T) = \sum_{n=0}^{\infty} \mathrm{e}^{-\beta\mathcal{E}(n,q)} \qquad \beta = \frac{1}{K_{\mathrm{B}}T}. \tag{13}$$

Then, the average energy density for this gas is defined in the usual way

$$\begin{aligned} U_{|q|}(\omega, T) &= \frac{8\pi}{c^3}\left(\frac{\omega}{2\pi}\right)^2\left[-\frac{\partial}{\partial\beta}\ln Z_q(\omega, T)\right] \\ &= \frac{8\pi}{c^3}\left(\frac{\omega}{2\pi}\right)^2 \frac{\sum_{n=0}^{\infty}\mathcal{E}(n,q)\mathrm{e}^{-\beta\mathcal{E}(n,q)}}{\sum_{n=0}^{\infty}\mathrm{e}^{-\beta\mathcal{E}(n,q)}}. \end{aligned} \tag{14}$$

For the case $q_2 = 0$ and $q_1 = 1$, we recover the classical Planck law

$$U_1(\omega, T) = \frac{8\pi}{c^3}\left(\frac{\omega}{2\pi}\right)^2 K_{\mathrm{B}}T\frac{\beta\hbar\omega}{\mathrm{e}^{\beta\hbar\omega} - 1}. \tag{15}$$

Furthermore, for $|q| \neq 1$, in dimensionless notation (with $x = \hbar\omega/(K_{\mathrm{B}}T)$), we get

$$Z_q(x) = \sum_{n=0}^{\infty}\exp\left\{-\frac{1}{2}x\left[\left(\frac{|q|^{2(n+1)} + |q|^{-2(n+1)} - 2\cos 2\theta(n+1)}{|q|^2 + |q|^{-2} - 2\cos 2\theta}\right)^{1/2} + \left(\frac{|q|^{2n} + |q|^{-2n} - 2\cos 2\theta n}{|q|^2 + |q|^{-2} - 2\cos 2\theta}\right)^{1/2} - 1\right]\right\} \tag{16}$$

$$u_q(x) = x^3\left[-\frac{\mathrm{d}}{\mathrm{d}x}\ln Z_q(x)\right] = x^3\frac{\sum_{n=0}^{\infty}\sigma(q,n)\mathrm{e}^{-(1/2)x\sigma(q,n)}}{\sum_{n=0}^{\infty}\mathrm{e}^{-(1/2)x\sigma(q,n)}} \tag{17}$$

where

$$\sigma(q,n) = \left(\frac{|q|^{2(n+1)} + |q|^{-2(n+1)} - 2\cos 2\theta(n+1)}{|q|^2 + |q|^{-2} - 2\cos 2\theta}\right)^{1/2} + \left(\frac{|q|^{2n} + |q|^{-2n} - 2\cos 2\theta n}{|q|^2 + |q|^{-2} - 2\cos 2\theta}\right)^{1/2} - 1. \tag{18}$$

Notice that for the above quantities, the symmetry $|q| \to |q|^{-1}$ is valid as it happens for real q.

We do not discuss the convergence of the series Z_q, in as much as this subject has already been studied by Martin-Delgado [3] with the result that the convergence is ensured for positive q (i.e. in our case the convergence is even stronger).

In the limiting case of high energies or low temperatures, i.e. $x = \hbar\omega/(K_B T) \gg 1$, the q-partition function $Z_q(x)$ can be well approximated by the first two terms of series (16)

$$\begin{aligned} Z_q(x) &\simeq 1 + \exp\left[-\frac{1}{2}x\left(\frac{|q|^4 + |q|^{-4} - 2\cos 4\theta}{|q|^2 + |q|^{-2} - 2\cos 2\theta}\right)^{1/2}\right] \\ &\simeq 1 + \exp\left[-\frac{1}{2}x(|q| + |q|^{-1})\left(1 - \frac{4\sin^2\theta}{(|q| + |q|^{-1})^2}\right)^{1/2}\right] \end{aligned} \tag{19}$$

or

$$Z_q(x) \simeq 1 + \exp\left[-\frac{x}{2}(|q| + |q|^{-1})\left(1 - 2\frac{\sin^2\theta}{(|q| + |q|^{-1})^2}\right)\right] \tag{20}$$

and for $|q| = e^\tau$, we obtain

$$Z_q(x) \simeq 1 + \exp\left[-x\cosh\tau\left(1 - \frac{\sin^2\theta}{2\cos^2 h\tau}\right)\right] \simeq 1 + \exp(-x\cosh\tau A) \equiv Z^w_{|q|}(x) \tag{21}$$

where

$$A = 1 - \frac{\sin^2\theta}{2\cosh^2\tau}. \tag{22}$$

Inserting the above approximation into (14), we obtain the $|q|$-extension of Wien law for the energy density $u^w_{|q|}(x)$

$$u^w_{|q|}(x) \simeq x^3 \cosh\tau A \exp(-x\cosh\tau A). \tag{23}$$

For real q, i.e. $\theta = 0$, we recover exactly the results of Martin-Delgado [3]. Moreover, from equation (23), it is easy to obtain the following extension of the Wien shift law for complex q

$$x_{\max} = 3 \Big/ \cosh\tau\left(1 - \frac{\sin^2\theta}{2\cos^2 h\tau}\right) \tag{24}$$

namely

$$\omega_{\max} = 3K_B T \Big/ \hbar\cosh\tau\left(1 - \frac{\sin^2\theta}{2\cosh^2\tau}\right) \tag{25}$$

which increases for complex q in analogy with the case of real q [3].

The $|q|$-Stefan law can be readily obtained by integrating the Wien approximation (23) of the energy density, thus obtaining

$$U_\tau^w(x) = \frac{6K_B^4}{\pi^2(\hbar\cosh\tau Ac)^3}T^4. \tag{26}$$

From (25), we see that the effect of the deformation in this approximation is easily understood by introducing an effective Planck constant $\hbar_\tau(\theta)$ as follows:

$$\hbar_\tau(\theta) = \hbar\cosh\tau\left(1-\frac{\sin^2\theta}{2\cosh^2\tau}\right). \tag{27}$$

The above effective Planck constant for $\theta = 0$, i.e. real q, coincides exactly with that given by Martin-Delgado [3], i.e.

$$\hbar_\tau \to \hbar\cosh\tau. \tag{28}$$

In summary, we have studied the Planck distribution for a complex q-boson gas by using a new q-boson oscillator algebra with complex deformation parameter [10]. The main advantage of such an algebra is that it allows us to get real (not complex [5]) number-operator and energy eigenvalues for the deformed harmonic oscillator. We obtain generalized results for the q-complex Planck distribution, which for real q coincides with those already given for a gas of q-bosons [3]. This constitutes a first application of the new q-complex boson algebra introduced in [10]. Other applications will be given elsewhere.

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